JOURNAL OF COMBINATORIAL THEORY, Series A 54, 123–128 (1990)

Note

An Inequality for Partially Ordered Sets

ISTVÁN BECK*

Department of Mathematics and Computer Sciences, University of Haifa, Haifa 31999, Israel

Communicated by the Managing Editors

Received April 27, 1988

INTRODUCTION

Ahlswede and Daykin [1] proved the following result: Let S be a distributive lattice, and let α, β, γ, and δ be non-negative real valued functions on S. If

$$\alpha(A)\,\beta(B) \leqslant \gamma(A \vee B)\,\delta(A \wedge B) \tag{1}$$

holds whenever A, B each contains exactly one point of S then (1) always holds, $(\alpha(A) = \sum \{\alpha(a) \mid a \in A\})$. Similar inequalities are discussed in the papers [2–8].

The object of this note is to consider a special case of this inequality and derive necessary and sufficient conditions for the equality to hold.

1. PRELIMINARIES

All posets in this note are finite. A subset I of a poset P is an *upper ideal* if for any $x \in I$ and any $y \in P$, $y \geqslant x \Rightarrow y \in I$, and a *lower* ideal is defined correspondingly. For a poset P let $\mathscr{I}d(P)$ denote the family of upper ideals in P. If we refer to an ideal without specifying whether it is upper or lower, we shall always intend an upper ideal. The elements x and y in a poset P are *path connected* if there exists a sequence of elements $x_1, ..., x_n$ in P such that $x_1 = x$, $x_n = y$ and x_i and x_{i+1} are comparable for each $1 \leqslant i \leqslant n-1$. A subset C *separates* the subsets A and B if any path from an element in

* Present address: H.S.R.–MAT/NAT, Box 2557 Ullandhaug, 4004 Stavanger, Norway.

0097-3165/90 $3.00

A to an element in B has to pass through C. The subsets A and B are said to be *separated* if they are not path connected.

We let $Id(x)$, $Id(x \,\&\, y)$, $Id(x \,\&\, \neg y)$ refer to the number of ideals (in P) that contain x, that contain both x and y, and the number of ideals that contain x and not y, respectively. We shall adopt a similar notation for any number of elements.

DEFINITION. The pair (P, f) is called a *real partially ordered set* if P is a partially ordered set and $f\colon P \to R^+$ is a function from P to the *positive* real numbers.

We shall usually assume that the function f is given, and simply say that P is a real poset. When (P, f) is a real poset, we let $f(I) = \prod_{x \in I} f(x)$ for any subset $I \subseteq P$, and $f(\varnothing) = 1$. For a subset $T \subseteq P$ define

$$Id^f(T) = \sum \{f(I) \mid I \text{ ideal and } T \subseteq I\}$$

and let

$$P^f(T) = \frac{Id^f(T)}{Id^f(\varnothing)}.$$

If the set T is a singleton $\{z\}$, we simply write $P^f(z)$ and $Id^f(z)$. We shall also use the notation $Id^f(x \,\&\, y)$, $Id^f(x \,\&\, \neg y)$, etc. In most of this paper we shall assume that the function $f\colon P \to R^+$ is given. Unless we are interested in some particular properties of the function f, the superscript in the expressions P^f and Id^f shall be omitted.

Since $f(I)\, f(J) = f(I \cap J)\, f(I \cup J)$ for any two ideals I and J of P, the result of Ahlswede and Daykin implies that

$$Id(A) \cdot Id(B) \leqslant Id(A \cup B) \cdot Id(A \cap B)$$

for any two subsets $A, B \subseteq P$. In order to clarify when the equality holds, it shall be necessary to give an independent proof of this inequality.

2. THE THEOREM

We shall prove

THEOREM 1. *Let A and B be subsets of a real poset (P, f). Then:*

$$Id^f(A \cup B) \cdot Id^f(A \cap B) \geqslant Id^f(A) \cdot Id^f(B)$$

and the equality holds exactly *when any path from A to B has to pass through the upper ideal generated by $A \cap B$.*

First an observation:

LEMMA 1. *Let I be an upper (lower) ideal of cardinality n. There exists then a chain of upper (lower) ideals*

$$I_1 \subset I_2 \subset \cdots \subset I_n = I$$

such that $|I_k| = k$.

Proof. Trivial.

We also observe:

LEMMA 2. *If Q is a lower ideal in P then $I = P - Q$ is an upper ideal, and there is a one-to-one correspondence between the upper ideals in P that contain I, and $\mathcal{I}d(Q)$. The mapping takes an upper ideal J that contains I to $J \cap Q \in \mathcal{I}d(Q)$. Since $f(A \cup B) = f(A) \cdot f(B)$ whenever A and B are disjoint, we derive that for any subset $T \subseteq Q$ one has*

$$Id_Q(T) = \frac{Id_P(T \cup I)}{f(I)}$$

and

$$Q(T) = \frac{P(T \cup I)}{P(I)}.$$

We shall first make some reductions of Theorem 1:

First reduction. We shall first demonstrate that it suffices to prove Theorem 1 in the case that A and B are disjoint. Assume that Theorem 1 has been verified in this case, and let A and B be any subsets in P. Let G be the upper ideal generated by $A \cap B$, and let $Q = P - G$. Let $A' = A - G$, and $B' = B - G$. Then A' and B' are disjoint subsets of the poset Q, and Theorem 1 may be applied and yields that

$$Id_Q(\varnothing) \cdot Id_Q(A' \cup B') \geqslant Id_Q(A') \cdot Id_Q(B')$$

and there is an equality when there is no path from A' to B' in Q. Combining this with Lemma 2 gives

$$\frac{Id(G)}{f(G)} \cdot \frac{Id(A' \cup B' \cup G)}{f(G)} \geqslant \frac{Id(A' \cup G)}{f(G)} \cdot \frac{Id(B' \cup G)}{f(G)}.$$

Since $Id(A' \cup B' \cup G) = Id(A \cup B)$, $Id(A' \cup G) = Id(A)$, $Id(B' \cup G) = Id(B)$ and $Id(G) = Id(A \cap B)$ the proof of this reduction is complete.

Second reduction. We shall then show that in order to prove Theorem 1 for disjoint subsets, it suffices to prove it when A and B are disjoint, and B is an upper ideal. Let A and B be arbitrary disjoint subsets of P. Let B^* be the upper ideal in P generated by B, and let $A' = A - B^*$. Then $A' \cap B^* = \emptyset$ and assuming Theorem 1 for this case yields that

$$Id(\emptyset) \cdot Id(A' \cup B^*) \geqslant Id(A') \cdot Id(B^*)$$

with equality iff A' and B^* are separated. Moreover, $Id(A' \cup B^*) = Id(A \cup B)$, $Id(B^*) = Id(B)$. Furthermore, $Id(A') \geqslant Id(A)$ and we have an equality iff the ideal generated by A' contains A. It is easily seen that A and B are separated exactly when A' and B^* are separated and the ideal generated by A' contains A. This proves the second reduction.

We shall now prove Theorem 1 in the case that A and B are disjoint, and B is an upper ideal.

LEMMA 3. *Let A be a subset of a real poset P. Then*

(1) *If Q is a lower ideal in P containing A, then $P(A) \leqslant Q(A)$.*

(2) *If Q is an upper ideal in P containing A, then $P(A) \geqslant Q(A)$.*

We have equality in (1) *or* (2) *iff A and $P - Q$ are separated.*

Before proving the lemma, let us note that Theorem 1 follows from Lemma 3.1 and the previous reductions. If A and B are disjoint subsets of P, and B is an upper ideal, we let $Q = P - B \supseteq A$. By Lemma 2,

$$Q(A) = \frac{P(A \cup B)}{P(B)}$$

and the proof of Theorem 1 is complete.

Proof of Lemma 3. The proof is by induction on the cardinality of the poset P. We shall only consider the case that A is a subset of a *lower* ideal Q of P. A very similar proof may be provided for the other case. Since $P(\emptyset) = 1$ for any poset P, we may assume that A is not empty. If $|P| \leqslant 2$ one easily proves the lemma by inspection. Assume now that both (1) and (2) of Lemma 3 has been proven for all posets with less than n elements and assume that $|P| = n$. Let Q be a lower ideal in P containing the set A. According to Lemma 1, it suffices to consider the case when $P = Q \cup \{z\}$, $z \notin Q$. Let L be the set of $y \in Q$ such that $z \geqslant y$. The set $U = Q - L$ is an upper ideal in Q. We divide the ideals in Q in two classes:

$\mathscr{A}$. The ideals in Q that are disjoint from L.

$\mathscr{B}$. The ideals in Q that are not disjoint from L.

Furthermore, let $\mathscr{A}^z$ denote the family of ideals $\{A \cup \{z\} \mid A \in \mathscr{A}\}$, and similarly define $\mathscr{B}^z$. Observe that $\mathscr{A} = \mathscr{I}d(U)$. Furthermore, $\mathscr{I}d(P)$ is the union of the disjoint families $\mathscr{A}$, $\mathscr{A}^z$, and $\mathscr{B}^z$. Recalling the multiplicative property of f on the disjoint union of two sets, we get the equations:

$$P(A) = \frac{\begin{pmatrix} 1+f(z)) \cdot \sum \{f(I) \mid I \in \mathscr{A} \;\&\; A \subseteq I\} \\ + f(z) \cdot \sum \{f(I) \mid I \in \mathscr{B} \;\&\; A \subseteq I\} \end{pmatrix}}{(1+f(z)) \cdot \sum \{f(I) \mid I \in \mathscr{A}\} + f(z) \cdot \sum \{f(I) \mid I \in \mathscr{B}\}}$$

$$Q(A) = \frac{\sum \{f(I) \mid I \in \mathscr{A} \;\&\; A \subseteq I\} + \sum \{f(I) \mid I \in \mathscr{B} \;\&\; A \subseteq I\}}{\sum \{f(I) \mid I \in \mathscr{A}\} + \sum \{f(I) \mid I \in \mathscr{B}\}}$$

$$P(z) = \frac{f(z) \cdot \sum \{f(I) \mid I \in \mathscr{A}\} + f(z) \cdot \sum \{f(I) \mid I \in \mathscr{B}\}}{(1+f(z)) \cdot \sum \{f(I) \mid I \in \mathscr{A}\} + f(z) \cdot \sum \{f(I) \mid I \in \mathscr{B}\}}.$$

If A is not contained in U, we let $U(A) = 0$, and if $A \subseteq U$ we have

$$U(A) = \frac{\sum \{f(I) \mid I \in \mathscr{A} \;\&\; A \subseteq I\}}{\sum \{f(I) \mid I \in \mathscr{A}\}}.$$

In both cases, simple calculation shows that

$$Q(A) - P(A) = \{Q(A) - U(A)\} \cdot \{1 - P(z)\}.$$

To complete the proof, let us first note that $P(z) < 1$ and $Q(A) > 0$. If A is *not* contained in U, then $U(A) = 0$, and it follows from the last equality that $Q(A) > P(A)$. This is the desired result, since in this case there exists a path from z to A.

If A is contained in U the induction hypothesis implies that $Q(A) \geqslant U(A)$ since U is an upper ideal in Q. Hence, $Q(A) \geqslant P(A)$. Moreover, there is a path from A to z iff there is a path from A to L. We again apply the induction hypothesis, and the proof of Lemma 3 is complete. This also completes the proof of Theorem 1.

COROLLARY 1. *Let (P, f) be a real poset, and let $x, y \in P$. Then*

$$Id^f(x \;\&\; y) \cdot Id^f(\neg x \;\&\; \neg y) \geqslant Id^f(x \;\&\; \neg y) \cdot Id^f(\neg x \;\&\; y)$$

and the equality holds only iff x and y are not path connected.

Proof. If $x = y$ the right side of the inequality becomes 0 while the left is positive. We assume that $x \neq y$. Theorem 1 applied to the sets $A = \{x\}$ and $B = \{y\}$ gives

$$Id^f(x \;\&\; y) \cdot Id^f(\varnothing) \geqslant Id^f(x) \cdot Id^f(y)$$

with equality when x and y are not path connected.

Using that $Id^f(\varnothing) = Id^f(x) + Id^f(\neg x)$, and $Id^f(y) = Id^f(x \,\&\, y) + Id^f(\neg x \,\&\, y)$ the inequality reduces to

$$Id^f(x \,\&\, y) \cdot Id^f(\neg x) \geqslant Id^f(x) \cdot Id^f(\neg x \,\&\, y).$$

Using that $Id^f(\neg x) = Id^f(\neg x \,\&\, y) + Id^f(\neg x \,\&\, \neg y)$ and the equality $Id^f(x) = Id^f(x \,\&\, y) + Id^f(x \,\&\, \neg y)$, we derive the desired result.

A particular case of the preceeding corollary is obtained when the function $f \equiv 1$. Then $Id(A)$ is the number of ideals containing the set A.

We shall now see what happens with the condition for equality in Theorem 1 if we relax the requirement that f is a function to the *positive* real numbers, but only assume f to be non-negative. We let $[A, B]^f = Id^f(A \cup B) \cdot Id^f(A \cap B) - Id^f(A) \cdot Id^f(B)$, and the final result in this note shows that the polynom $[A, B]^f$ supplies useful information about the cut sets of P. The corollary has some flavour of a Nullstellensatz.

COROLLARY 2. *Let A, B be disjoint subsets of the partially ordered set P and let f be a non-negative real valued function on P. Then $[A, B]^f = 0$ iff $Id^f(A) \cdot Id^f(B) = 0$, or if every path from A to B passes through the lower ideal generated by the zeroes of f.*

Proof. Let $K = \{x \in P \mid f(x) = 0\}$, and let K^* denote the lower ideal generated by K. We note that if U is an upper ideal in P then $f(U) = 0 \Leftrightarrow U \cap K \neq \varnothing \Leftrightarrow U \cap K^* \neq \varnothing$. It is now easily seen that $Id^f(A) = 0 \Leftrightarrow A \cap K^* \neq \varnothing$. Hence it suffices to prove the equivalence when both A and B are contained in the upper ideal $Q = P - K^*$. Moreover, $\mathscr{I}d(Q)$ coincides with the upper ideals of P that are contained in Q. Since $f(U) = 0$ for any upper ideal not contained in Q, it follows that $[A, B]^f_P = [A, B]^f_Q$. Corollary 2 follows now from Theorem 1.

REFERENCES

1. R. AHLSWEDE AND D. E. DAYKIN, An inequality for the weights of two families of sets, their unions and intersections, *Z. Wahrsch. Verw. Gebiete* **43** (1978), 183–185.
2. R. AHLSWEDE AND D. E. DAYKIN, Inequalities for a pair of maps $S \times S \rightarrow S$ with S a finite set, *Math. Z.* **165** (1979), 267–289.
3. I. ANDERSON, Intersection theorems and lemma of Kleitman, *Discrete Math.* **16** (1976), 181–185.
4. D. E. DAYKIN, A lattice is distributive iff $|A|\,|B| \leqslant |A \vee B|\,|A \wedge B|$, *Nanta Math.* **10** (1977), 58–60.
5. D. E. DAYKIN, D. J. KLEITMAN, AND D. B. WEST, The number of meets between two subsets of a lattice, *J. Combin. Theory* **1** (1979), 135–156.
6. C. M. FORTUIN, P. W. KASTELEYN, AND J. GINIBRE, Correlation inequalities on some partially ordered sets, *Comm. Math. Phys.* **22** (1971), 89–103.
7. R. HOLLEY, Remarks on the FKG inequalities, *Comm. Math. Phys.* **36** (1974), 227–231.
8. D. J. KLEITMAN, Families of non-disjoint subsets, *J. Combin. Theory* **1** (1966), 153–155.